Stochastic Deformations of Sample Paths of Random Walks and Exclusion Models

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Abstract: This study in centered on models accounting for stochastic deformations of sample paths of random walks, embedded either in $\mathbb{Z}^2$ or in $\mathbb{Z}^3$. These models are immersed in multi-type particle systems with exclusion. Starting from examples, we give necessary and sufficient conditions for the underlying Markov processes to be reversible, in which case their invariant measure has a Gibbs form. Letting the size of the sample path increase, we find the convenient scalings bringing to light phase transition phenomena. Stable and metastable configurations are bound to time-periods of limiting deterministic trajectories which are solution of nonlinear differential systems: in the example of the ABC model, a system of Lotka-Volterra class is obtained, and the periods involve elliptic, hyper-elliptic or more general functions. Lastly, we discuss briefly the contour of a general approach allowing to tackle the transient regime via differential equations of Burgers' type.

Key-words: Random walk, deformation, exclusion process, thermodynamic limit, phase transition, Burgers' equation.

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Déformations stochastiques de trajectoires de marches aléatoires et modèles d'exclusion

Résumé : Cette étude a pour objet de rendre compte de l'évolution stochastique d'échantillons de marches aléatoire plongées dans $\mathbb{Z}^2$ ou $\mathbb{Z}^3$. Une formulation en termes de réactions-diffusions unidimensionnelles, de particules avec exclusion est proposée. A partir d'exemples, des conditions nécessaires et suffisantes de réversibilité du processus de Markov sous-jacent sont établies, conduisant à une mesure de Gibbs. En faisant croître la taille des échantillons vers l'infini, nous trouvons un changement d'échelle approprié mettant en évidence des phénomènes de transition de phase. Les états stables et metastables font intervenir des trajectoires déterministes, solutions de systèmes différentiels non-linéaires, de type Lotka-Volterra dans les cas les plus simples. Les contours d'une approche générale, permettant d'englober le régime transitoire via un système différentiel d'équations de type Burgers, sont évoqués en conclusion.

Mots-clés : Marche aléatoire, déformations processus d'exclusion, limite thermodynamique, transition de phase, équation de Burgers.

1 Introduction

We are interested in models describing evolution of sample paths of random walks, when they are submitted to random local deformations involving possibly several links. Roughly speaking, given a finite sample path, say of size N, forming a not necessarily closed curve, the problem will be to characterize the evolution of an associated family $\{\mathbb{Y}_i, i = 1, \ldots, N\}$ of Markov processes in the thermodynamic limit as $N \to \infty$. This requires to guess and to find the interesting scalings.

In a previous study [7], we considered random walks on a square lattice, deformations involved pairs of links and occurred at the epoches of Poisson jump processes in continuous time (see section 2 for a more exact definition). The analysis was carried out by means of an explicit mapping, which led to view the system as a coupling of two exclusion processes.

Starting from a number of observations, we intend to hint in this paper that the model in [7] can indeed be cast into a broader class, the ultimate goal being to propose methods of wide applicability concerning the following questions:

- conditions ensuring Gibbs states and explicit forms of the corresponding invariant measures;
- steady-state equations in the thermodynamic limit as $N \to \infty$, and their solutions in the case of Gibbs states, but also in situations involving permanent currents;
- hydrodynamic and transient equations, when N is sufficiently large, yielding thus a complete picture of the evolution.

Generalizations of the model in $\mathbb{Z}^2$ can follow two natural trends. First, in modifying the construction of the random walk. Indeed, in the square lattice, we dealt with a 4-letter alphabet. Considering instead a finite alphabet of l letters is then tantamount to constructing random walks with oriented links, whose affixes are multiples of $\frac{2k\pi}{l}, k = 0, \ldots, l-1$. The case $l = 2$ corresponds to the simple exclusion process in $\mathbb{Z}$, and $l = 3$ yields the so-called *ABC model*.

Another possible extension is to relax the constraint that the walk lives in $\mathbb{Z}^2$ and to define a stochastic deformation process in higher dimension.

In the sequel, we shall restrict ourselves to some paradigms in $\mathbb{Z}^2$ and $\mathbb{Z}^3$. In section 2, we define a class of two-dimensional models, together with related patterns in $\mathbb{Z}^3$,

in terms of exclusion particle systems. Section 3 is devoted to stochastic reversibility of the Markov processes of interest and to the Gibbs form of their invariant measure. In section 4, the non-symmetric classical ABC model is solved (fundamental scaling, phase transitions, classification of stable configurations) through the analysis of a Lotka-Volterra differential system. The concluding section 5 gives a brief overview of ongoing research about large scale dynamics, nonequilibrium and transient regimes.

2 Model descriptions via exclusion particle systems

Our main objective in this section is to show how the evolution of the sample paths of random walks can be fruitfully described by means of particle exclusion processes.

Beforehand, to avoid repetition and clumsy notation, let us emphasize that we shall only deal with jump Markov processes in continuous time. So implicitly the word *transition rate* will always refer to some underlying generator. Also, N will always stand for the size of the sample path, or equivalently the number of its links.

2.1 Preliminaries

In $\mathbb{Z}^1$, the simple exclusion model coincides with the well known KPZ system (see e.g. [9]), which represents a fluctuating and eventually growing interface. This system is coded by a sequence of binary variables $\{\tau_j\}$, $j = 1, \ldots, N$, depending on whether a particle is present or not, with asymmetric jump rates. This system has been extensively studied. In particular, the invariant measure has been obtained in a matrix solution form, for fairly arbitrary parameters and boundary conditions [3]. Large scale dynamics has also been analyzed [12, 14], showing Burgers' equations [1].

2.2 2-dimensional models

1) The triangular lattice and the ABC model

Here the evolution of the random walk is restricted to the triangular lattice. Each link (or step) of the walk is either 1, $e^{2i\pi/3}$ or $e^{4i\pi/3}$, and quite naturally will be said to be of type A, B and C, respectively. This corresponds to the so-called *ABC model*,

since there is a coding by a 3-letter alphabet. The set of *transitions* (or reactions) is given by

$$AB \underset{p^+}{\overset{p^-}{\leftrightarrows}} BA, \qquad BC \underset{q^+}{\overset{q^-}{\leftrightarrows}} CB, \qquad CA \underset{r^+}{\overset{r^-}{\leftrightarrows}} AC, \tag{2.1}$$

where we have introduced general rates. Also we impose *periodic boundary conditions* on the sample paths. This model was first introduced in [4] in the context of particles with exclusion, and a Gibbs form corresponding to reversibility has been found in [5] in some cases.

2) The square lattice and coupled exclusion model.

This model was introduced in [7] to analyze stochastic deformations of a walk in the square lattice, and it will be referred to from now on as the $\{\tau_a \tau_b\}$ model. Assuming links counterclockwise oriented, we the following transitions can take place.

$$AB \underset{\lambda^{ba}}{\overset{\lambda^{ab}}{\rightleftarrows}} BA, \qquad BC \underset{\lambda^{cb}}{\overset{\lambda^{bc}}{\rightleftarrows}} CB, \qquad CD \underset{\lambda^{dc}}{\overset{\lambda^{cd}}{\rightleftarrows}} DC, \qquad DA \underset{\lambda^{ad}}{\overset{\lambda^{da}}{\rightleftarrows}} AD,$$

$$AC \underset{\delta^{bd}}{\overset{\gamma^{ac}}{\rightleftarrows}} BD, \qquad BD \underset{\delta^{ca}}{\overset{\gamma^{bd}}{\rightleftarrows}} CA, \qquad CA \underset{\delta^{db}}{\overset{\gamma^{ca}}{\rightleftarrows}} DB, \qquad DB \underset{\delta^{ac}}{\overset{\gamma^{db}}{\rightleftarrows}} AC.$$

We studied a rotation invariant version of this model when

$$\begin{cases} \lambda^+ \overset{\text{def}}{=} \lambda^{ab} = \lambda^{bc} = \lambda^{cd} = \lambda^{da}, \\ \lambda^- \overset{\text{def}}{=} \lambda^{ba} = \lambda^{cb} = \lambda^{dc} = \lambda^{ad}, \\ \gamma^+ \overset{\text{def}}{=} \gamma^{ac} = \gamma^{bd} = \gamma^{ca} = \gamma^{db}. \\ \gamma^- \overset{\text{def}}{=} \delta^{ac} = \delta^{bd} = \delta^{ca} = \delta^{db}. \end{cases}$$

Define the mapping $(A, B, C, D) \to (\tau^a, \tau^b) \in \{0,1\}^2$, such that

$$\begin{cases} A \to (0,0), \\ B \to (1,0), \\ C \to (1,1), \\ D \to (0,1). \end{cases}$$

Then the dynamics can be formulated in terms of coupled exclusion processes. The evolution of the sample path is represented by a Markov process with state space the

$2N$ binary random variables $\{\tau_j^a\}$ and $\{\tau_j^b\}$, $j = 1, \ldots, N$, taking value 1 if a particle is present and 0 otherwise. The jump rates to the right $(+)$ or to the left $(-)$ are given by

$$\begin{cases} \lambda_a^{\pm}(i) = \bar{\tau}_i^b \bar{\tau}_{i+1}^b \lambda^{\mp} + \tau_i^b \tau_{i+1}^b \lambda^{\pm} + \bar{\tau}_i^b \tau_{i+1}^b \gamma^{\mp} + \tau_i^b \bar{\tau}_{i+1}^b \gamma^{\pm}, \\ \lambda_b^{\pm}(i) = \bar{\tau}_i^a \bar{\tau}_{i+1}^a \lambda^{\pm} + \tau_i^a \tau_{i+1}^a \lambda^{\mp} + \bar{\tau}_i^a \tau_{i+1}^a \gamma^{\pm} + \tau_i^a \bar{\tau}_{i+1}^a \gamma^{\mp}. \end{cases} \tag{2.2}$$

Notably, one sees the jump rates of a given sequence are locally conditionally defined by the complementary sequence.

3) An extended stochastic clock model

We propose an extension of the preceding model for an arbitrary two-dimensional regular graph. To this end, consider a random walk composed of oriented links, the affixes of which take values $\omega_k = \exp(2ik\pi/n), k = 1, \ldots, n$, the n-th roots of unity. There are two different situations.

(a) $n = 2p + 1$ is odd. Then the walk cannot have a *fold* of two successive links, so that local displacements of edges can only be performed by exchanging two successive links. Let $X = \{X^1, \ldots, X^n\}$, denote the particle types viewed as letters of an alphabet, and let $\{\lambda^{kl}\}$ be the transition rates. The set of reactions is defined by

$$X^k X^l \underset{\lambda^{lk}}{\overset{\lambda^{kl}}{\rightleftarrows}} X^l X^k, \quad k \in [1,\, 2p+1],\, k \neq l.$$

These rules provide an extension of the ABC model, which we shall discuss in detail in section 3.

(b) When $n = 2p$ is even, the grammar is altered when two successive links fold, so that this elementary transition amounts merely to a rotation of the fold of angle $\pm\frac{2\pi}{n}$ (instead of a π rotation which would occur when exchanging the two links). The situation is thus slightly more complicated and the set of reactions is now given by

$$\begin{cases} X^k X^l \underset{\lambda^{lk}}{\overset{\lambda^{kl}}{\rightleftarrows}} X^l X^k, & k = 1, \ldots, n, \quad l \neq k + p, \\ X^k X^{k+p} \underset{\delta^{k+1}}{\overset{\gamma^k}{\rightleftarrows}} X^{k+1} X^{k+p+1}, & k = 1, \ldots, n, \end{cases} \tag{2.3}$$

where $k + p$ is taken modulo n. It is worth noting that γ^k (resp. δ^k) concern folds rotating in the counterclockwise direction (resp. clockwise) and that the number of

letters of each type is no longer conserved. In other words, odd models give rise to pure diffusions with eventual drifts, when even models are truly reaction-diffusion models.

2.3 3-dimensional generalizations of the 2-coupled exclusion model on a diamond lattice

2.3.1 Elementary deformations

Actually, the diamond lattice formulation of stochastic deformations in $\mathbb{Z}^3$ provides several straightforward generalization of the 2-dimensional $\{\tau_a\tau_b\}$ model. Indeed, between two nodes there are $8 = 2^3$ possible links (the jumps of the sample path). Let (τ_a, τ_b, τ_c) be the vector of binary components corresponding to a displacement in each direction, where a, b and c denote here the three particle families (letters).

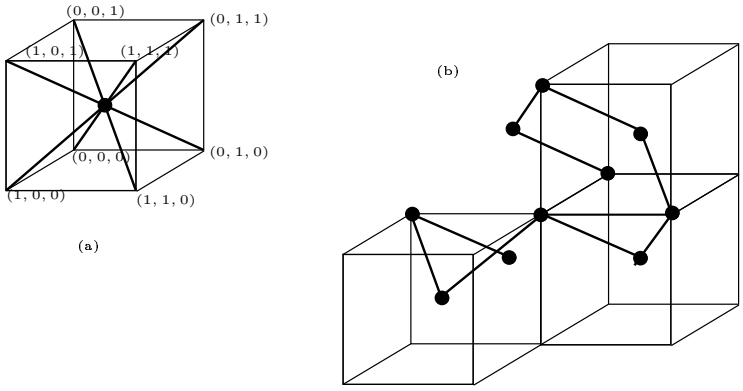
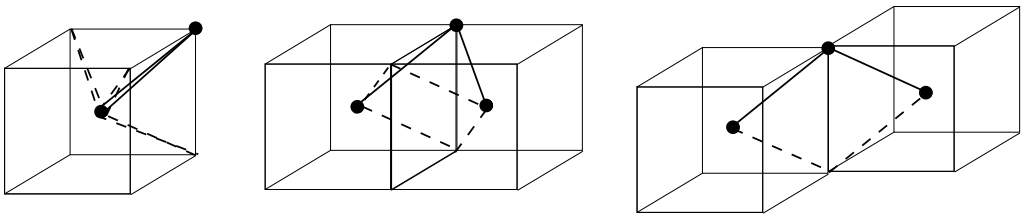


Fig. 2.1: 3-binary coding of links of the walks on a diamond lattice

As in the 2-dimensional model, elementary deformations consist in exchanging between neighbouring sites the value of one of the binary components. By construction this model gives a kind of geometric decoupling between the three types of particles. In fact all possible existing stochastic coupling result solely from the conditional transition rates $\lambda_a^{\pm}(\tau_b, \tau_c)$, $\lambda_b^{\pm}(\tau_c, \tau_a)$, and $\lambda_c^{\pm}(\tau_a, \tau_b)$, according to the various possible models. Geometrically the corresponding mutations for the 3-dimensional walk are depicted in figure 2.2.



(a) (b) (c)

Fig. 2.2: Various elementary transformations depending on the patterns formed by two successive links. Initial (resp. final) configurations in plain (resp. dotted) lines: fold (a), bend (b), twisted bend (c).

In order to obtain some non-trivial dynamical effect, we couple the three systems of particles, keeping in mind the possibility of getting back the 2-dimensional $\{\tau_a\tau_b\}$ model under certain conditions. This is the subject of the next two paragraphs.

2.3.2 A linear coupled exclusion model

Here we propose a coupling which is linear with respect to the complementary fields, via the following intensities:

$$\begin{cases} \lambda_a^{\pm}(i) = \lambda \pm 2(\tau_i^b - \tau_i^c)\mu_a, \\ \lambda_b^{\pm}(i) = \lambda \pm 2(\tau_i^c - \tau_i^a)\mu_b, \\ \lambda_c^{\pm}(i) = \lambda \pm 2(\tau_i^a - \tau_i^b)\mu_c. \end{cases}$$

Suppose for a while $\mu_c = 0$ and $\mu_a = \mu_b = \mu$. Then the sequence $\{tau_c\}$ remains disordered, which means the marginal law of τ_i^c is $\frac{1}{2}$. As for the subsystem $\{\tau_a, \tau_b\}$, up to random contributions $(\tau_i^c - \frac{1}{2})\mu$, the transition rates $\lambda_a^{\pm}$ and $\lambda_b^{\pm}$ correspond to a particular definition of the $\{\tau_a\tau_b\}$ model. Hence, in the limit, we obtain a kind of *disordered* $\{\tau_a\tau_b\}$ model.

2.3.3 A non-linear coupled exclusion model

It is also expected to recover the $\tau_a\tau_b$ model when one of the components is completely frozen in an ordered phase, taking for example $\tau_i^c = 0$ for $i = 1, \ldots, N$. This situation is fulfilled by choosing the following non-linear couplings

$$\begin{cases} \lambda_a^{\pm}(i) = \lambda \pm (\tau_i^b - \bar{\tau}_i^b)(\tau_i^c - \bar{\tau}_i^c)\mu_a, \\ \lambda_b^{\pm}(i) = \lambda \pm (\tau_i^c - \bar{\tau}_i^c)(\tau_i^a - \bar{\tau}_i^a)\mu_b, \\ \lambda_c^{\pm}(i) = \lambda \pm (\tau_i^a - \bar{\tau}_i^a)(\tau_i^b - \bar{\tau}_i^b)\mu_c. \end{cases}$$

2.4 Boundary conditions

Although they live in $\mathbb{Z}^2$ or $\mathbb{Z}^3$, let us emphasize that all the objects considered throughout this study are curves, hence one-dimensional dynamical systems.

Hence, for any given sample path of size N, there are two links referred to as site 1 and N, at which *boundary conditions* have to be specified. We shall consider only *periodic* boundary conditions: this means essentially the system is *invariant* under circular permutation of the sites. Consequently, certain geometric quantities locally conserved will remain also globally conserved. For example, the distance between the two extremities (not necessarily distinct) of a curve remains constant, so that closed curves will stay closed for ever.

3 Reversibility and Gibbs measures

In the sequel, our goal will be to find exact scalings permitting to derive phase transition conditions, for deformation processes of the class defined in section 2, as $N \to \infty$.

In fact, for the sake of shortness, we shall restrict ourselves to case studies where the sample paths have a *Gibbs invariant measure*, but this is not a «sine qua non» condition as commented in section 5.

3.1 Equilibrium states

We shall establish conditions, which are either of geometrical nature or bear directly on the transition rates, to ensure the processes of interest are *reversible*: detailed balance equations hold and they suffice to equilibrate all possible cycles in the state space (see e.g. [10]). In that case a potential does exist and the invariant measure can be expressed explicitly as a Gibbs measure.

3.1.1 Odd alphabet

For 2-dimensional walks, when the size n of the alphabet is odd, as already noticed in section 2.2, there is a dual point of view saying that a n-species particle system moves in a one-dimensional lattice: there is exactly one particle per site and the transition rates λ^{kl} correspond to exchanges of a particle k with a particle l between adjacent sites. In a very different context, this model was proposed in [5], from which we extract some results related to our topic.

Up to a slight abuse in the notation, we let $X_i^k \in \{0,1\}$ denote the binary random variable representing the occupation of site i by a letter of type k. The state of the system is represented by the array $\mathbb{X} \stackrel{\text{def}}{=} \{X_i^k, i = 1,\dots,N; k = 1,\dots,n\}$ of size $N \times n$. Then the invariant measure of the associated Markov process is given by

$$P(\mathbb{X}) = \frac{1}{Z} \exp\big[-\mathcal{H}(\mathbb{X})\big], \tag{3.1}$$

where

$$\mathcal{H}(\mathbb{X}) = \sum_{i<j} \sum_{k,l} \alpha^{kl} X_i^k X_j^l, \tag{3.2}$$

and

$$\alpha^{kl} - \alpha^{lk} = \log \frac{\lambda^{kl}}{\lambda^{lk}},$$

provided that the following condition holds

$$\sum_{k \neq l} \alpha^{kl} N_k = 0. \tag{3.3}$$

Indeed, a typical balance equation reads

$$\frac{P[\dots, X_i^k = 1, X_{i+1}^l = 1, \dots]}{P[\dots, X_i^l = 1, X_{i+1}^k = 1, \dots]} = \frac{\lambda^{lk}}{\lambda^{kl}} = \exp(\alpha^{lk} - \alpha^{kl}), \tag{3.4}$$

and relation (3.3) proceeds directly from enforcing the above measure to be invariant by translation.

3.1.2 Even alphabet

When the cardinal of the alphabet is even, say $n \stackrel{\text{def}}{=} 2p$, the situation is rendered a bit more involved due possible rotations of consecutive folded links. There is no longer conservation law for each letter number, and one should instead introduce the quantities

$$\Delta^k \stackrel{\text{def}}{=} N^{k+p} - N^k, \quad k = 1, \ldots, p-1,$$

which represent the differences between populations of links with opposite directions. Moreover, as a rule, some non-trivial cycles in the state-space are not balanced (see figure 3.3), unless transition rates satisfy additional conditions. This gives rise to the next theorem.

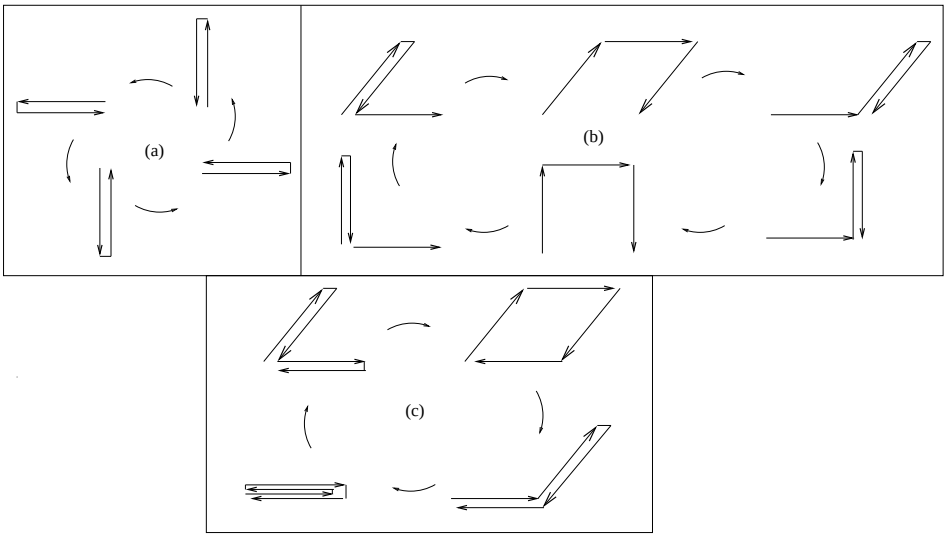


Fig. 3.3: Elementary cycles: fold (a), 3-link motion (b), square loop (c).

Theorem 3.1. *Assume $n = 2p$ and periodic boundary conditions. Then the system is* reversible *if and only if the following conditions are imposed on the rates and on*

the particles numbers:

$$(i) \quad \prod_{k=l}^{l+p-1} \frac{\gamma^k}{\delta^{k+1}} = 1, \forall l = 1, \ldots, n$$

$$(ii) \quad \frac{\lambda^{kl}}{\lambda^{lk}} \frac{\lambda^{k+p,\,l}}{\lambda^{l,\,k+p}} = 1, \quad \forall k, l = 1 \ldots n,\ k \neq l,$$

$$(iii) \quad \sum_{l \neq k+p} \Delta^l \log \frac{\lambda^{kl}}{\lambda^{lk}} = 0, \quad k = 1, \ldots, n.$$

The result relies on the next lemma.

Lemma 3.2. *In the case of periodic boundary conditions, if the invariant measure has a Gibbs form given by (3.1) and (3.2), then the following relationships must hold:*

$$(iv) \quad \begin{cases} \alpha^{kl} - \alpha^{lk} = \log \dfrac{\lambda^{kl}}{\lambda^{lk}}, \quad k = 1, \ldots, n,\ l \neq k+p\,; \\ \alpha^{k+1,\,k+p+1} - \alpha^{k,\,k+p} = \log \dfrac{\gamma^k}{\delta^{k+1}}, \quad k = 1, \ldots, n\,; \end{cases}$$

(v) *there exists a constant* $\alpha \in \mathbb{R}$ *such that*
$\alpha^{kl} + \alpha^{k+p,l} = \alpha^{lk} + \alpha^{l,k+p} = \alpha,\ \forall k, l = 1, \ldots, n\,;$

$$(vi) \quad \sum_{l=1}^{p-1} (\alpha^{kl} - \alpha^{lk}) \Delta^l = 0, \quad k = 1, \ldots, n.$$

Proof. We only present the main steps. Condition (iv) in the lemma comes from a balance equation of type (3.4). The case $l = k + p$ corresponds to adjacent folded links, and, after setting

$$U(k,i) \stackrel{\text{def}}{=} \sum_l \sum_{j>i+1} \big(\alpha^{k+1,\,l} + \alpha^{k+p+1,\,l} \ - \ \alpha^{kl} - \alpha^{k+p,\,l}\big) X_j^l,$$

$$V(k,i) \stackrel{\text{def}}{=} \sum_l \sum_{j<i} (\alpha^{l,\,k+1} + \alpha^{l,\,k+p+1} \ - \ \alpha^{lk} - \alpha^{l,\,k+p}) X_j^l,$$

equation (3.4) has to be replaced by

$$\frac{P[\ldots, X_i^k = 1, X_{i+1}^{k+p} = 1, \ldots]}{P[\ldots, X_i^{k+1} = 1, X_{i+1}^{k+p+1} = 1, \ldots]} = \frac{\gamma^k}{\delta^{k+1}}$$

$$= \exp\big[\alpha^{k+p+1,\,k+1} - \alpha^{k,\,k+p} - U(k,i) - V(k,i)\big],$$

which leads to impose condition (v).

To take into account the invariance by translation, let σ denote a circular permutation σ among the indices, such that $\sigma(i) = 1 + i \mod (N)$. Consider

$$\mathcal{H}_\sigma(\mathbb{X}) = \sum_{1 \geq k,l \geq N} \sum_{i<j} \alpha^{kl} X^k_{\sigma(i)} X^l_{\sigma(j)},$$

the resulting energy obtained after applying permutation σ. Then

$$\mathcal{H}_\sigma(\mathbb{X}) = \mathcal{H}(\mathbb{X}) + \sum_{k=1}^{N} X_1^k \sum_{l=1}^{N} (\alpha^{kl} - \alpha^{lk}) N^l,$$

where N^l is the number of links l. Since $N^{l+p} - N^l$, $l = 1, \ldots, p$ is conserved (but N^l is not), the rule (v) leads to the translational invariance in the form of (vi).

As to reversibility, one can check that (i) and (ii) are necessary to equilibrate the cycles depicted in figure 3.3. Moreover, the cycle condition imposed by a circular permutation is exactly given by (iii). Indeed, this cycle is performed by transporting one particle through the system from site 1 to site N: during this operation, a tagged particle X^k will encounter the N^l particles corresponding to all other species $l \neq k$ and the resulting transition weight is then given by $\displaystyle\prod_{l \neq k,k+p} \left[\frac{\lambda^{kl}}{\lambda^{lk}}\right]^{N^l}$, which in turn, by using (ii), amounts to condition (iii) after taking the logarithm. These three conditions are in fact sufficient to determine the parameters $\{\alpha^{kl}\}$ in order to solve $(iv), (v), (vi)$, thus ensuring reversibility.

■

4 Steady state of the ABC system in the thermodynamic limit

As an illustration, we will present a detailed analysis of the thermodynamic equilibrium situation in the case of the ABC model.

With three particles species, the form $\mathcal{H}$ of (3.2) comes to

$$\mathcal{H}(\{\mathbb{X}\}) = \sum_{i<j} \alpha^{ab} A_i B_j + \alpha^{bc} B_i C_j + \alpha^{ca} C_i A_j, \tag{4.1}$$

where the constants $\alpha^{ab}, \alpha^{bc}, \alpha^{ca}$ take the values

$$\alpha^{ab} = \log \frac{p^+}{p^-}, \quad \alpha^{bc} = \log \frac{q^+}{q^-}, \quad \alpha^{ca} = \log \frac{r^+}{r^-},$$

and the constraints (3.3) now become

$$\frac{N_A}{N_B} = \frac{\alpha^{bc}}{\alpha^{ca}}, \quad \frac{N_B}{N_C} = \frac{\alpha^{ca}}{\alpha^{ab}}, \quad \frac{N_C}{N_A} = \frac{\alpha^{ab}}{\alpha^{bc}}. \tag{4.2}$$

4.1 Scaling and Lotka-Volterra equations

In the example of the ABC model [2], we have at hand an explicit analytic expression for the invariant measure. In fact, our claim is that equations in the thermodynamic limit can be derived by using a method proposed in [7] for the square lattice model, which a priori *does not require any explicit knowledge* of the invariant measure, which is most of the time untractable. The approach consists in conditioning the motion of a tagged particle with respect to admissible configurations. We sketch without proof the main lines of argument in the case of the ABC model.

For the sake of shortness, (A, B, C) will denote the $3N$-dimensional bolean vector $\{(A_i, B_i, C_i), i = 1, \ldots, N\}$.

Consider the steady state behaviour as $t \to \infty$ and let q_i^a denote the random variable representing the conditional probability of finding a particle A_i on site i, given (B, C). It is possible, although difficult, to show that

$$P(A \mid B, C) = \prod_{i=1}^{N} \big[A_i q_i^a + \overline{A}_i (1 - q_i^a)\big],$$

where the conditional equilibrium of type A particles satisfies

$$\frac{q_{i+1}^a}{q_i^a} = \frac{\lambda_a^+(i)}{\lambda_a^-(i+1)},$$

with conditional rates given by

$$\begin{cases} \lambda_a^+(i) = p^+ B_i + r^- C_i + \Gamma\, \overline{B}_i \overline{C}_i, \\ \lambda_a^-(i+1) = p^- B_i + r^+ C_i + \Gamma\, \overline{B}_i \overline{C}_i, \end{cases}$$

these expressions being similar to the form of (2.2) in the $\{\tau_a \tau_b\}$ model. Γ is a non-zero quantity taking into account the exclusion at site i in the following sense: when $\overline{B}_i \overline{C}_i = 1$ then necessarily $A_i = 1$. The proof relies on the convergence of an iterative rocking-scheme.

By the boolean nature of B_i and C_i, can write

$$\log \frac{q_{i+1}^a}{q_i^a} = \alpha^{ca} C_i \, - \, \alpha^{ab} B_i, \tag{4.3}$$

with two analogous equations for B and C species.

Fundamental scaling We introduce the so-called *fundamental scaling* defined by

$$\alpha^{bc} = \frac{\alpha}{N} + o\Big(\frac{1}{N}\Big), \quad \alpha^{ca} = \frac{\beta}{N} + o\Big(\frac{1}{N}\Big), \quad \alpha^{ab} = \frac{\gamma}{N} + o\Big(\frac{1}{N}\Big),$$

where α, β, γ are three positive real constant.

Limiting equations Setting from now on $x \stackrel{\text{def}}{=} i/N$, for $1 \leq i \leq N$, one can show that the weak limits

$$\rho^a(x) = \lim_{N\to\infty} q_{xN}^a, \qquad \rho^b(x) = \lim_{N\to\infty} q_{xN}^b, \qquad \rho^c(x) = \lim_{N\to\infty} q_{xN}^c$$

exist and satisfy the system of deterministic differential equations

$$\begin{cases} \dfrac{\partial \rho_a}{\partial x} = \rho_a(\beta\rho_c - \gamma\rho_b), \\ \dfrac{\partial \rho_b}{\partial x} = \rho_b(\gamma\rho_a - \alpha\rho_c), \\ \dfrac{\partial \rho_c}{\partial x} = \rho_c(\alpha\rho_b - \beta\rho_a), \end{cases} \tag{4.4}$$

with the crucial constraints due to the periodic boundary conditions

$$\rho_u(x+1) = \rho_u(x), \quad \forall u = a, b, c. \tag{4.5}$$

Proof. Starting from (4.3), one applies the law of large numbers and ergodic theorems to justify the approximation of finite sums by Riemann integrals, as $N \to \infty$.

■

It is amusing to see that (4.4) belongs to the class of generalized Lotka-Volterra systems. The original Lotka-Volterra model was the simplest model of predator-prey interactions, proposed independently by Lotka (1925) and Volterra (1926), see for instance [13]. Nonetheless, in our case the world is less crual and all particle types are on an equal footing...!

The form of (4.4) lends itself to a solution in terms of special functions. The first step is to remark the existence of two level surfaces

$$\rho_a + \rho_b + \rho_c = 1, \tag{4.6}$$

$$\rho_a^\alpha \rho_b^\beta \rho_c^\gamma = \kappa, \tag{4.7}$$

where (4.6) follows at once from (4.2) and κ is a constant of motion to be determined. Using (4.6) to eliminate ρ_c, we rewrite (4.7) as

$$\rho_a^\alpha \rho_b^\beta (1 - \rho_a - \rho_b)^\gamma = \kappa. \tag{4.8}$$

The change of functions $u = \rho_a + \rho_b$ and $v = \beta\rho_a - \alpha\rho_b$, yields the first order nonlinear differential equation

$$\frac{du}{dx} = (1-u)v(u) \tag{4.9}$$

where $v(u)$ satisfies the equation

$$(\alpha u + v)^\alpha (\beta u - v)^\beta (1-u)^\gamma = \kappa(\alpha+\beta)^{\alpha+\beta} \tag{4.10}$$

Formally, $u(x)$ can be expressed as

$$x = \int_{u_(0)}^{u(x)} \frac{du}{(1-u)v(u)}. \tag{4.11}$$

It appears that the ratios in (4.2) are rational for all finite N, but, since we have let $N, N_A, N_B, \to \infty$, they might become arbitary real numbers in the interval $[0,1]$. When they are rational, (4.9) is a polynomial equation, and then $u(x)$ is in some

sense just a bit more general that an hyperelliptic function, since, after some algebra, we are left with integrals of the form

$$\int \sqrt{\left[s^2 - a^2(1-s)^{-\frac{p}{q}}\right]} ds,$$

where p and q stand for positive integers. The particular case $\alpha = \beta = \gamma$ is rather simple, since then

$$\left(\frac{du}{dx}\right)^2 = (1-u)[u^2(1-u) - \kappa],$$

showing u to be a standard Jacobi elliptic function.

For any given κ, the system (4.4) admits of a unique solution. In particular, there is *always* a degenerate solution crumbled to the fixed point

$$\widetilde{\rho}_a = \frac{\alpha}{\alpha+\beta+\gamma}, \quad \widetilde{\rho}_b = \frac{\beta}{\alpha+\beta+\gamma}, \quad \widetilde{\rho}_c = \frac{\gamma}{\alpha+\beta+\gamma}, \tag{4.12}$$

and corresponding to the constant

$$\widetilde{\kappa} \stackrel{\text{def}}{=} \frac{\alpha^\alpha \beta^\beta \gamma^\gamma}{(\alpha+\beta+\gamma)^{\alpha+\beta+\gamma}}. \tag{4.13}$$

The purpose of the next paragraph is to discriminate between solutions of (4.4) and (4.5), in order to relate them to admissible limit-points of Φ_N, as $N \to \infty$.

4.2 Stability, fundamental period and phase transition

To catch a rough qualitative insight into the solution of (4.4), a standard approach relies on a linearization of the right-hand side around the fixed point (4.12). This yields a linear differential system, whose matrix

$$\frac{1}{\alpha+\beta+\gamma} \begin{bmatrix} 0 & -\alpha\gamma & \alpha\beta \\ \beta\gamma & 0 & -\alpha\beta \\ -\beta\gamma & \alpha\gamma & 0 \end{bmatrix},$$

has three eigenvalues 0 and $\lambda_\pm = \pm i\sqrt{\alpha\beta\gamma(\alpha+\beta+\gamma)}$, and the trajectories are located on an ellipsoid. But, since the above eigenvalues are purely imaginary, it is well known that no conclusion can be drawn as for the original system, which might be of a quite different nature.

However, it is pleasant to see that the modulus of the nonzero eigenvalues plays in fact a crucial role as shown in the next theorem.

Theorem 4.1. *Let* $s \stackrel{\text{def}}{=} \alpha+\beta+\gamma$ *and* $\eta \stackrel{\text{def}}{=} \frac{s}{3}$. *The limit* $\Phi \stackrel{\text{def}}{=} \lim_{N\to\infty} \Phi_N$ *of the ABC model is deterministic and there is a second order phase transition phenomenon. There exits a critical value*

$$\eta_c \stackrel{\text{def}}{=} \frac{2\pi}{3\sqrt{\widetilde{\rho}_a\widetilde{\rho}_b\widetilde{\rho}_c}},$$

such that if $\eta > \eta_c$ *then there are closed non-degenerate trajectories of (4.4) satisfying (4.5), with period* $T(\kappa_p) = \frac{1}{p}$, $p \in \{1, \ldots, [\frac{\eta}{\eta_c}]\}$. *The only admissible stable* Φ *corresponds*

- *either to the trajectory associated with* κ_1 *if* $\eta > \eta_c$;
- *or to the degenerate one consisting of the single point (4.12) if* $\eta \le \eta_c$.

The proof involves a forest of technicalities and we only sketch the main lines of argument. The first step is to switch to polar coordinates

$$\begin{cases} u_a \stackrel{\text{def}}{=} \widetilde{\rho}_a - \dfrac{\alpha}{s} = r\cos\theta \\ u_b \stackrel{\text{def}}{=} \widetilde{\rho}_b - \dfrac{\alpha}{s} = r\sin\theta. \end{cases}$$

Rewrite (4.7) as

$$H(r,\theta) = \log\kappa, \tag{4.14}$$

with

$$H(r,\theta) \stackrel{\text{def}}{=} \alpha\log\big[r\cos\theta + \frac{\alpha}{s}\big] + \beta\log\big[r\sin\theta + \frac{\beta}{s}\big] + \gamma\log\big[\frac{\gamma}{s} - r(\cos\theta+\sin\theta)\big],$$

and let $r(\theta,\kappa)$ be the single root in r of (4.14). Then θ satisfies the differential equation

$$\frac{d\theta}{dx} = G(\theta,\kappa), \tag{4.15}$$

where

$$\begin{aligned} G(\theta,\kappa) &\stackrel{\text{def}}{=} \frac{1}{s}\big[\beta(\alpha+\gamma)\cos\theta + \alpha\beta\sin^2\theta + \alpha(\beta+\gamma)\sin^2\theta\big] \\ &+ r(\theta,\kappa)\cos\theta\sin\theta\big[(\beta+2\alpha+2\gamma)\cos\theta + (\alpha+2\beta+2\gamma)\sin\theta\big]. \end{aligned}$$

Letting $T(\kappa)$ be the period of the orbit, we have

$$T(\kappa) = \int_0^{2\pi} \frac{d\theta}{G(\theta,\kappa)}.$$

The second important step relies on the monotonic behaviour of $T(\kappa)$ with respect to the parameter κ, yielding the inequality

$$T(\kappa) \geq T(\widetilde{\kappa}).$$

Observing that

$$r(\theta, \widetilde{\kappa}) = 0, \quad \forall\, \theta \in [0, 2\pi],$$

we can write, by (4.15), $T(\widetilde{\kappa})$ as a contour integral on the unit circle, namely

$$T(\widetilde{\kappa}) = -4is \oint_{\Gamma} \frac{1}{z^2[\gamma(\beta-\alpha) - 2i\alpha\beta] + 2z[2\alpha\beta + \gamma(\alpha+\beta)] + \gamma(\beta-\alpha) + 2i\alpha\beta},$$

or, after a simple calculus,

$$T(\widetilde{\kappa}) = 2\pi\sqrt{\frac{s}{\alpha\beta\gamma}},$$

which leads precisely to the critical value η_c announced in the theorem.

5 Perspectives

This paper is the continuation of [7], but is certainly an intermediate step. For the sake of shortness, we did restrict ourselves to the thermodynamics of the ABC model. Actually, our goal is to analyze the dynamics of random curves evolving in $\mathbb{Z}^m$ (no spatial constraints) or in $\mathbb{Z}_+^m$, when they warp under the action of some stochastic deformation grammar.

In [6], this project will be carried out in the framework of large scale dynamics for exclusion processes, and it will mainly address the points listed hereafter.

More on thermodynamic equilibrium The trick to derive limiting differential systems amounts essentially to writing conditional flow equations on suitable sample paths, even in the presence of particle currents. These equations involve functionals of Markov and they enjoy special features encountered in many systems. It might also be interesting to note that most of the Lotka-Volterra equations can be explained in the light of the famous urns of Ehrenfest.

Phase transition There exists a global interpretation by means of a free energy functional with two components: the entropy of the system, and the algebraic area enclosed by the curve. It turns out that the contention between these two quantities

yield, after taking limits $\lim_t \to \infty \lim_N \to \infty$ (in that order), either stretched deterministic curves or Brownian objects when the scaling is of central limit-type.

Transient regime Our claim is that time-dependent behaviour can be treated along the same ideas, up to technical subtleties, by means of a numerical scheme based on the conservation of particle currents. This should yield a system of Burgers equations, extending those obtained in [7] for the symmetric $\{\tau_a\tau_b\}$ model, which had the form

$$\begin{cases} \dfrac{\partial\rho^a(x,t)}{\partial t} &= D\dfrac{\partial^2\rho^a(x,t)}{\partial x^2} - 2D\eta\dfrac{\partial}{\partial x}\big[\rho^a(1-\rho^a)(1-2\rho^b)\big](x,t), \\ \dfrac{\partial\rho^b(x,t)}{\partial t} &= D\dfrac{\partial^2\rho^b(x,t)}{\partial x^2} + 2D\eta\dfrac{\partial}{\partial x}\big[\rho^b(1-\rho^b)(1-2\rho^a)\big](x,t). \end{cases}$$

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